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# Random Vibrations of Nonlinear Continua Endowed with Fractional Derivative Elements

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## Abstract

In this paper, two techniques are proposed for determining the large displacement statistics of random exciting continua endowed with fractional derivative elements: Boundary Element Method (BEM) based Monte Carlo simulation; and Statistical Linearization (SL). The techniques are applied to the problem of nonlinear beam and plate random response determination in the case of colored random external load. The BEM is implemented in conjunction with a Newmark scheme for estimating the system response in the time domain in conjunction with repeated simulations, while SL is used for estimating efficiently and directly, albeit iteratively, the response statistics.

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*Keywords:* continuum mechanics; fractional derivative; boundary element method; statistical linearization.

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## 1. Introduction

Fractional operators have received recently considerable attention in a number of thematic domains [1]. In this regard, a notable example pertains to viscoelasticity as discussed in the works, for instance, of Nutting [2], Gemant [3], Scott-Blair and Gaffyn [4], and Bagley and Torvik [5]. The impact of fractional operators in problems involving the vibration of systems endowed with fractional derivative elements was described in two extensive review articles by Rossikhin and Shitikova [6, 7] in a deterministic setting, while analyses focusing on systems exposed to random loads are more recent. In this context, Spanos and Zeldin [8] elucidated certain pitfalls associated with the use of frequency dependent parameters in conjunction with the calculation of system response statistics. Linear systems

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were investigated by Agrawal [9], and Di Paola, Failla, et al. [10], while nonlinear systems were investigated by techniques such as SL, stochastic averaging, Weiner path integrals and harmonic wavelet based SL [11-15]. The vibration problem of continuous systems has received some attention, as well. In this context, most articles deal with the analysis of either linear or nonlinear beams [16-19], and of plates [20-22]. Despite these efforts, the need of techniques for estimating the nonlinear response of continuous systems endowed with fractional derivative elements reliably and efficiently persists. Thus, this paper describes a solution technique based on the SL technique that allows estimating efficiently the response statistics of continuous systems and, further, a strictly numerical approach based on the BEM, for conducting Monte Carlo studies. The techniques are applied to the specific problems of large beam and plate displacements.

## 2. Governing equations

### 2.1. Preliminary concepts on fractional derivatives

A critical concept underlining the definition of fractional derivative relates to the definition of fractional integral, which is obtained as the convolution of a function  $w(t)$  with a power law kernel. That is,

$${}_0D_t^{-\alpha} = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{w(\tau)}{(t-\tau)^{-\alpha+1}} d\tau, \text{ for } \alpha > 0, \quad (1)$$

with  $\Gamma(\alpha)$  being the gamma function [23]. Clearly, for integer values of the power law,  $\alpha = n$ , the gamma function renders the factorial of the integer number and, thus eq. (1) provides the classical  $n$ -fold integral. The Riemann-Liouville (RL) fractional derivative is constructed by differentiating eq. (1)  $m$  times. That is,

$${}_0^{RL}D_t^\alpha = \frac{1}{\Gamma(m-\alpha)} \frac{d^m}{dt^m} \int_0^t \frac{w(\tau)}{(t-\tau)^{\alpha+1-m}} d\tau, \quad m-1 \leq \alpha < m. \quad (2)$$

Further, the Grünwald-Letnikov (GL) representation [24] is given by the equation

$${}_0^{GL}D_t^\alpha w(t) = \sum_{k=0}^{m-1} \frac{w^{(k)}(0)t^{k-\alpha}}{\Gamma(k+1-\alpha)} + \frac{1}{\Gamma(m-\alpha)} \int_0^t \frac{w^{(m)}(\tau)}{(t-\tau)^{\alpha+1-m}} d\tau, \quad m-1 < \alpha < m. \quad (3)$$

Such a representation can lead to algorithms for the numerical computation of fractional derivatives. Indeed, the series in eq. (3) can be expanded and the series representation of the GL derivative

$${}_0^{GL}D_t^\alpha w(t) = \lim_{\Delta t \rightarrow 0} \Delta t^{-\alpha} \sum_{k=0}^n GL_k w(t-k\Delta t), \quad (4)$$

can be derived, where  $GL_k$  are calculated recursively using the relationship

$$GL_k = \frac{k-\alpha-1}{k} GL_{k-1}; \quad GL_0 = 1. \quad (5)$$

Eq. (4) provides the G1-algorithm that is used in this paper for conducting requisite numerical integrations. Such an algorithm clearly points out the fading memory property of the fractional derivative.

## 2.2. Equations governing the large vibration of a continuous system

Continuous systems are described via partial differential equations of the form

$$D[v] = q, \quad (6)$$

where  $v$  is the unknown system response,  $D[\cdot]$  is a nonlinear differential operator, and  $q$  is the load. Developing an exact solution to problem (6) is a daunting task. Thus, two approximate techniques are proposed herein: Boundary Element Method (BEM) based Monte Carlo simulations; and Statistical Linearization (SL). These techniques can be utilized for estimating the system response when the nonlinear operator  $D[\cdot]$  includes a fractional derivative element.

The BEM is implemented in conjunction with the nonlinear problem (6) by resorting to a surrogate linear problem accommodating the numerical implementation of the BEM as commonly proposed in the literature. Specifically, an appropriate linear partial differential equation of the form

$$L[v] = b, \quad (7)$$

where  $L[\cdot]$  is a linear operator and  $b$  is a fictitious time-dependent term, is selected. Such a problem admits an integral representation of the solution, which is conveniently discretized to determine the system response as

$$v = \mathbf{G} \cdot \mathbf{b}, \quad (8)$$

with  $\mathbf{G}$  being a known matrix, and  $\mathbf{b}$  being an unknown (time-dependent) vector. Finally, the unknown source term  $\mathbf{b}$  is calculated by collocating eq. (6) in each element of the continuum, so that the equation

$$\mathbf{B}[\mathbf{b}] = \mathbf{q}. \quad (9)$$

is obtained. Eq. (9) is a system of nonlinear ordinary differential equations including the fractional derivative operator that can be solved in the time domain via a numerical integration scheme. Thus, the source term  $\mathbf{b}$  is calculated by integrating eq. (9) in the time domain and the system response is readily computed via eq. (8) at each time step. Next, this procedure is repeated a large number of times to derive Monte Carlo results for the continuum response statistics.

SL involves the replacement of a nonlinear differential equation by an equivalent linear system in which the system parameters are determined by minimizing a mean square error between the nonlinear equation and the linear one. Its application in conjunction with continua is based on a prior representation of the system response via Galerkin expansion. Such an expansion is used for deriving the system of nonlinear ordinary differential equations

$$D_{nl}[w] = Q, \quad (10)$$

which is successively replaced by the equivalent linear system,

$$D_{eq}[w] = Q, \quad (11)$$

whose parameters are determined by minimizing the mean square error

$$\varepsilon = E\{(D_{nl}[w] - D_{eq}[w])^2\}. \quad (12)$$

These preceding procedures which here have been outlined succinctly, will be elucidated in the next sections, by considering specific problems pertaining to nonlinear random vibrations of a beam and of a plate.

### 3. Nonlinear vibration of a beam

The equation governing the large deflection  $v(x,t)$  of a vibrating beam of length  $L$  is

$$EI \frac{\partial^4 v(x,t)}{\partial x^4} + \rho A \frac{\partial^2 v(x,t)}{\partial t^2} + c_0 \partial_t^\alpha v(x,t) - N \frac{\partial^2 v(x,t)}{\partial x^2} = q(x,t), \quad (13)$$

where the symbols  $E$ ,  $I$ ,  $\rho$ , and  $A$  denote elastic modulus, moment of inertia of the cross-section, mass density, and cross-sectional area, respectively. Further,  $c$  and  $\alpha$  are parameters of a fractional derivative element;  $q(x,t)$  is a stochastic load, and  $N$  is the axial force given by the equation

$$N = \frac{EA}{2L} \int_0^L \left( \frac{\partial v(x,t)}{\partial x} \right)^2 dx. \quad (14)$$

The load is assumed of the separable form  $q(x,t) = p(x)f(t)$ , where  $p(x)$  is a deterministic function, and  $f(t)$  is a random Gaussian process of given power spectral density function  $S(\omega)$ .

#### 3.1. BEM solution

The BEM solution is sought by utilizing the equations

$$\frac{\partial^4 v(x,t)}{\partial x^4} = b(x,t). \quad (15)$$

In this context, the response representation is

$$v = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \int_0^L G(x,\xi) b(\xi,t) d\xi, \quad (16)$$

where  $G = |x-\xi|(x-\xi)^2/12$  and the constants  $c_i$  ( $i=0,\dots,3$ ) are dependent on the specific boundary conditions.

By discretizing the beam into  $N$  elements, assuming that the source term  $b(\xi,t)$  is constant over each element, and collocating eq. (13) in each nodal point, the set of equations

$$\rho A \underline{\underline{G}} \cdot \ddot{\underline{b}}(t) + c \underline{\underline{G}}_0 D_t^\alpha \underline{b}(t) + EI \underline{b}(t) - \frac{EA}{2L} \underline{F}(\underline{b}(t), \underline{\underline{G}}) = \underline{q}(t), \quad (17)$$

is obtained, where  $\underline{b}(t)$  is a vector containing the nodal values of the fictitious load,  $\underline{\underline{G}}$  is a known matrix arising from the integration of  $G(x,\xi)$  on the elements,  $\underline{F}(\underline{b}(t), \underline{\underline{G}})$  is a nonlinear vector function and  $\underline{q}(t)$  is a vector containing the load time histories.

Eq. (17) is integrated numerically. For this purpose, the fractional derivative element is approximated by the representation (4). In this sense, the known past values of  $\underline{b}(t)$  are used for estimating an effecting excitation accounting for the influence of the past terms. This approach allows deriving the incremental equation of motion

$$\rho A \underline{\underline{G}} \cdot \Delta \ddot{\underline{b}}(t_i) + c \Delta t^{-\alpha} \underline{\underline{G}} \cdot \Delta \underline{b}(t_i) + EI \Delta \underline{b}(t_i) - \frac{EA}{2L} \Delta \underline{F}(t_i) = \Delta \underline{q}(t_i) - c \Delta t^{-\alpha} \underline{\underline{G}} \cdot \left( \sum_{k=1}^{i-1} GL_k \Delta \underline{b}(t_i - k\Delta t) + GL_i \underline{b}(0) \right), \quad (18)$$

which is implemented by a classical Newmark scheme. From a computational perspective, it is worth - mentioning that a truncation can be introduced by neglecting the small contributions provided by past terms associated with  $k$  larger than a certain threshold.

### 3.2. Statistical linearization solution

The vertical displacement of the beam is represented by the equation,

$$v(x, t) = \sum_{m=1}^{\infty} w_m(t) \Phi_m(x), \quad (19)$$

where  $\Phi_m(x)$  are the orthogonal linear modes of beam vibration, and  $w_m(t)$  are time-dependent functions. By introducing the quantities

$$K_{mn} = K_{nm} = \int_0^L \Phi'_m \Phi'_n dx, \text{ and } R_{mn} = \int_0^L \Phi_m \Phi''_n dx, \quad (20)$$

and by algebraic manipulations reflecting error projection on the space of the beam modes, a nonlinear ordinary differential equation describing the time variation of  $w_m(t)$  is derived. Specifically:

$$\ddot{w}_m + \frac{c}{\rho A} D_t^\alpha w_m + \omega_m^2 w_m - \frac{E}{2\rho L^2} \sum_n \sum_i \sum_j w_n w_i w_j K_{ij} R_{mn} = \frac{P_m}{\rho AL} f(t), \text{ for } m = 1, 2, \dots, \quad (21)$$

where  $\omega_m$  is the natural frequency of vibration and

$$P_m = \int_0^L p(x) \Phi_m(x) dx. \quad (22)$$

Next, an approximate solution of eq. (21) is sought by replacing it by the system,

$$\ddot{w}_m + \frac{c}{\rho A} D_t^\alpha w_m + \omega_{eq,m}^2 w_m = \frac{P_m}{\rho AL} f(t), \text{ for } m = 1, 2, \dots, \quad (23)$$

in which  $\omega_{eq,m}$  is an optimal equivalent stiffness. In this context, the procedure for determining the equivalent linear stiffness leads to the equation

$$\omega_{eq,m}^2 = \omega_m^2 - \frac{E}{2\rho^3 A^2 L^4} \frac{1}{P_m S_{mm}} \sum_n \sum_i \sum_j K_{ij} R_{mn} P_n P_i P_j (S_{mn} S_{ij} + S_{mi} S_{nj} + S_{mj} S_{ni}), \text{ for } m=1, 2, \dots \quad (24)$$

where

$$S_{mn} = \int_{-\infty}^{+\infty} H_m(\omega) S(\omega) H_n(-\omega) d\omega, \quad (25)$$

and  $H_m(\omega)$  is the system transfer function

$$H_m(\omega) = [-\omega^2 + c(i\omega)^\alpha / (\rho A) + \omega_{eq,m}^2]^{-1}. \quad (26)$$

The numerical calculation of the equivalent stiffness is pursued iteratively by assuming at the first iteration that  $\omega_{eq,m} = \omega_m$  in the right hand side of eq. (24). Then, the second-order statistics of the response is estimated via the equivalent linear system:

$$\sigma^2(x) = \langle v^2(x,t) \rangle = (\rho AL)^{-2} \int_{-\infty}^{+\infty} S(\omega) \sum_m \sum_n \Phi_m(x) \Phi_n(x) P_m P_n H_m(\omega) H_n(-\omega) d\omega. \quad (27)$$

#### 4. Nonlinear vibration of a plate

Consider, the transverse displacement  $u = u(x,y,t)$  of a rectangular plate of sides  $a$  and  $b$ , with mass density  $\rho$ , thickness  $h$ , Young modulus  $E$ , and flexural stiffness  $D$  which is subject to a transverse load  $q = q(x,y,t)$  dependent on the space coordinates  $(x,y)$  and on the time variable  $t$  and is endowed with a fractional derivative element of order  $\alpha$  and constant damping  $c$ . The associated equation of motion is

$$\rho h \frac{\partial^2 u}{\partial t^2} + c_0 \partial_t^\alpha u + D \nabla^4 u - h \left( \frac{\partial^2 \phi}{\partial y^2} \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 \phi}{\partial x^2} \frac{\partial^2 u}{\partial y^2} - 2 \frac{\partial^2 \phi}{\partial x \partial y} \frac{\partial^2 u}{\partial x \partial y} \right) = q, \quad (28)$$

with  $\nabla^4 = (\partial^4 / \partial x^4 + \partial^4 / \partial y^4 + 2 \partial^4 / \partial x^2 \partial y^2)$  being the biharmonic operator, and  $\phi = \phi(x,y,t)$  being the Airy stress function that is governed by the equation

$$\nabla^4 \phi = E \left[ \left( \frac{\partial^2 u}{\partial x \partial y} \right)^2 - \frac{\partial^2 u}{\partial x^2} \frac{\partial^2 u}{\partial y^2} \right]. \quad (29)$$

The load is assumed of a separable type, so that  $q(x,y,t) = p(x,y)f(t)$ , where  $p(x,y)$  is a deterministic function and  $f(t)$  is a random process of a given power spectral density function  $S(\omega)$ .

##### 4.1. BEM solution

In this context, the BEM based numerical approach considers the two linear equations

$$\nabla^4 u = b_1(x,y,t), \text{ and } \nabla^4 \phi = b_2(x,y,t), \quad (30)$$

where  $b_1(x,y,t)$  and  $b_2(x,y,t)$  are space-time dependent fictitious loads.

The solution of the problems (30) has the integral representation [25]

$$\varepsilon u(P) = \int_A \Lambda_4 b_1 dA - \int_\Gamma \Lambda_1 u + \Lambda_2 u_n + \Lambda_3 \nabla^2 u + \Lambda_4 (\nabla^2 u)_n ds, \quad (31)$$

where  $\varepsilon = 2\pi$  or  $\pi$  if the point  $P$  is inside the domain  $A$  or on the boundary  $\Gamma$  respectively, and the other quantities are given in Ref. [25].

Further, the equation

$$\varepsilon \nabla^2 u(P) = \int_A \Lambda_2 b_1 dA - \int_\Gamma \Lambda_1 \nabla^2 u + \Lambda_2 (\nabla^2 u)_n ds. \quad (32)$$

holds. Eq. (31) and (32) are used for estimating the unknown boundary quantities by introducing the associated boundary conditions. For this purpose, the plate domain and boundary are discretized and eq. (31) and (32) are

collocated on boundary points. By doing so, the linear system of equations

$$\begin{bmatrix} [A_{11}] & [A_{12}] & [0] & [A_{14}] \\ [A_{21}] & [A_{22}] & [A_{23}] & [0] \\ [A_{31}] & [A_{32}] & [A_{33}] & [A_{34}] \\ [0] & [0] & [A_{43}] & [A_{44}] \end{bmatrix} \begin{Bmatrix} \{u\} \\ \{u_n\} \\ \{\nabla^2 u\} \\ \{(\nabla^2 u)_n\} \end{Bmatrix} = \begin{Bmatrix} \{B_1\} \\ \{B_2\} \\ \{0\} \\ \{0\} \end{Bmatrix} + \begin{bmatrix} 0 \\ 0 \\ [C_3] \\ [C_4] \end{bmatrix} [b_1]. \quad (33)$$

is derived. The sub-matrices composing the first two rows are determined by the boundary conditions. System (33) allows determining the boundary quantities in terms of the fictitious load  $b_1$ . Thus, the response of the plate is calculated by the equation

$$\underline{u} = \underline{G}_1 \underline{b}_1, \quad (34)$$

where  $\underline{G}_1$  is a known matrix,  $\underline{b}_1$  is a vector containing the values of the fictitious load at each point of the domain and  $\underline{u}$  is a vector containing the response at that points. A similar procedure can be used for representing the stress function:

$$\phi = \underline{G}_2 \underline{b}_2, \quad (35)$$

where it is observed that the only difference with the determination of  $\underline{u}$  relates to the different boundary conditions. The representation obtained in this manner is used for collocating the displacements and stress values into the original equations (28) - (29) in order to derive a set of fractional nonlinear ordinary differential equations for the fictitious loads  $\underline{b}_1$  and  $\underline{b}_2$ . That is,

$$\rho h \underline{G}_1 \ddot{\underline{b}}_1 + c \underline{G}_1 {}^0 D_t^\alpha \underline{b}_1 + D \underline{b}_1 - F_1(G_{=1,xx}, G_{=2,xx}, G_{=1,yy}, G_{=2,yy}, G_{=1,xy}, G_{=2,xy}) = \underline{q}, \text{ and } \underline{b}_2 = EF_2(G_{=1,xx}, G_{=1,yy}, G_{=1,xy}), \quad (36)$$

where  $\underline{F}_1$  and  $\underline{F}_2$  are nonlinear functions encapsulating the nonlinear elements of the original system.

The numerical solution of this fractional differential equation is obtained by a Newmark based algorithm by utilizing the same approach described for the beam vibration problem. The resulting incremental equation of motion is

$$\rho h \underline{G}_1 \cdot \Delta \ddot{\underline{b}}_1(t_i) + c \Delta t^{-\alpha} \underline{G}_1 \cdot \Delta \underline{b}_1(t_i) + D \Delta \underline{b}_1(t_i) - \Delta F_1(t_i) = \Delta \underline{q}(t_i) - c \Delta t^{-\alpha} \underline{G}_1 \cdot \left( \sum_{k=1}^{i-1} GL_k \Delta \underline{b}_1(t_i - k \Delta t) + GL_i \underline{b}_1(0) \right). \quad (37)$$

#### 4.2. Statistical linearization solution

The response of the system is represented by Galerkin expansions of the vertical displacement and of the stress function having time-dependent coefficients. Specifically,

$$u = \sum_{m,n} w_{mn}(t) U_{mn}(x, y), \text{ and } \phi = \frac{P_x y^2}{2bh} + \frac{P_y x^2}{2ah} + \sum_{m,n} w_{mn}^{(2)}(t) \varphi_{mn}(x, y), \quad (38)$$

with  $P_x$  and  $P_y$  being the total tension loads applied on the sides  $x=0, a$  and  $y=0, b$  of the plate, respectively, the eigen-functions  $U_{mn}$  and  $\varphi_{mn}$  depend upon the specific boundary conditions, and are orthogonal to each another. Further, the quantities

$$R_{xx}(M, N, m, n) = \iint_A \frac{\partial^2 U_{mn}}{\partial x^2} U_{MN} dA \text{ and } R_{yy}(M, N, m, n) = \iint_A \frac{\partial^2 U_{mn}}{\partial y^2} U_{MN} dA. \quad (39)$$

are introduced. By substituting eq. (38) into eq. (28) and (29), doing algebraic manipulations reflecting error projection in the space of eigen-functions, and observing that the stress function amplitudes  $w_{mn}^{(2)}$  can be expressed in terms of  $w_{mn}$ , a nonlinear fractional ordinary differential equation for the time-dependent amplitudes  $w_{mn}$  is found. Specifically,

$$\ddot{w}_{MN} + \frac{c}{\rho h} D_t^\alpha w_{MN} + \omega_{MN}^2 w_{MN} - \frac{4}{ab\rho h} \left( \frac{P_x}{b} \sum_{m,n} w_{mn} R_{xx}(M, N, m, n) + \frac{P_y}{a} \sum_{m,n} w_{mn} R_{yy}(M, N, m, n) \right) - \frac{4}{ab} \frac{E}{\rho} \sum_{m,n} \sum_{k,l} \sum_{p,q} w_{mn} w_{kl} w_{pq} I(M, N, m, n, k, l, p, q) = \frac{4}{ab\rho h} P_{MN} f(t), \quad \text{for } M, N = 1, 2, \dots \quad (40)$$

where

$$I(M, N, m, n, k, l, p, q) = \sum_{i,j} \iint_A \frac{\partial^2 U_{mn}}{\partial x^2} \frac{\partial^2 \varphi_{ij}}{\partial y^2} U_{MN} + \frac{\partial^2 U_{mn}}{\partial y^2} \frac{\partial^2 \varphi_{ij}}{\partial x^2} U_{MN} - 2 \frac{\partial^2 U_{mn}}{\partial x \partial y} \frac{\partial^2 \varphi_{ij}}{\partial x \partial y} U_{MN} dA \times \frac{\iint_A \frac{\partial^2 U_{kl}}{\partial x \partial y} \frac{\partial^2 U_{pq}}{\partial x \partial y} \varphi_{ij} - \frac{\partial^2 U_{kl}}{\partial x^2} \frac{\partial^2 U_{pq}}{\partial y^2} \varphi_{ij} dA}{\iint_A \frac{\partial^4 \varphi_{ij}}{\partial x^2 \partial y^2} \varphi_{ij} + \frac{\partial^2 \varphi_{ij}}{\partial x^2 \partial y^2} \varphi_{ij} + 2 \frac{\partial^2 \varphi_{ij}}{\partial x \partial y} \varphi_{ij} dA}, \quad (41)$$

$$P_{MN} = \iint_A p(x, y) U_{MN} dA, \quad (42)$$

and  $\omega_{MN}$  denotes the natural frequency of the linear plate. An approximate solution of eq. (40) is sought by replacing this nonlinear system by the equivalent linear system

$$\ddot{w}_{MN} + c/(\rho h) D_t^\alpha w_{MN} + \omega_{eq,MN}^2 w_{MN} = 4 P_{MN} f(t) / (\rho h a b), \text{ for } M, N = 1, 2, \dots \quad (43)$$

In this context, the equivalent stiffness is determined using the equation

$$\omega_{eq,MN}^2 = \omega_{MN}^2 - \frac{4}{ab\rho h} \frac{1}{P_{MN} S_{MN,MN}} \sum_{m,n} P_{mn} S_{MN,mn} \left[ \frac{P_x}{b} R_{xx}(M, N, m, n) + \frac{P_y}{a} R_{yy}(M, N, m, n) \right] - \left( \frac{4}{ab} \right)^3 \frac{E}{\rho^3 h^2} \frac{1}{P_{MN} S_{MN,MN}} \sum_{m,n} \sum_{k,l} \sum_{p,q} P_{mn} P_{kl} P_{pq} (S_{MN,mn} S_{kl,pq} + S_{MN,kl} S_{mn,pq} + S_{MN,pq} S_{mn,kl}) \times I(M, N, m, n, k, l, p, q); \quad \text{for } M, N = 1, 2, \dots, \quad (44)$$

where

$$S_{MN,mn} = \int_{-\infty}^{+\infty} H_{MN}(-\omega) S(\omega) H_{mn}(\omega) d\omega, \quad (45)$$



and  $H_{MN}(\omega)$  is the system transfer function

$$H_{MN}(\omega) = \frac{1}{-\omega^2 + c(i\omega)^\alpha / (\rho h) + \omega_{eq,MN}^2}. \quad (46)$$

Next, the standard deviation of the transverse displacement can be determined by the equation

$$\sigma^2(x, y) \approx u^2(x, y) \approx \left( \frac{4}{ab\rho h} \right)^2 \sum_{m,n} \sum_{k,l} P_{mn} P_{kl} U_{mn} U_{kl} S_{mn,kl}. \quad (47)$$

## 5. Concluding remarks

The random response of nonlinear continua endowed with fractional elements has been determined approximately by Boundary Element Method (BEM) based Monte Carlo simulations, and by Statistical Linearization (SL). The techniques have been described in conjunction with two structural dynamic problems: a nonlinear beam vibration and a nonlinear plate vibration. Figure 1 and 2 show typical representations of the response statistics obtained for a simply supported and for a rectangular plate. It is seen that SL is not only an efficient but also a reliable technique for calculating the response statistics.

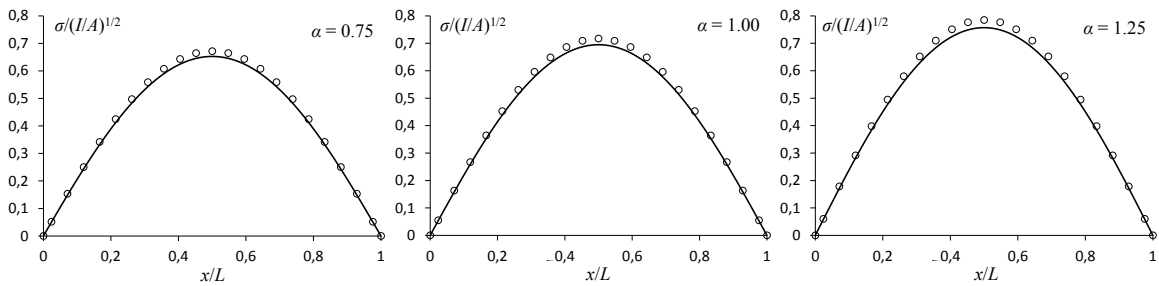


Fig. 1. Standard deviation of the vertical displacement of a simply supported beam for various fractional derivative orders. Continuous lines: SL; dotted lines: BEM.

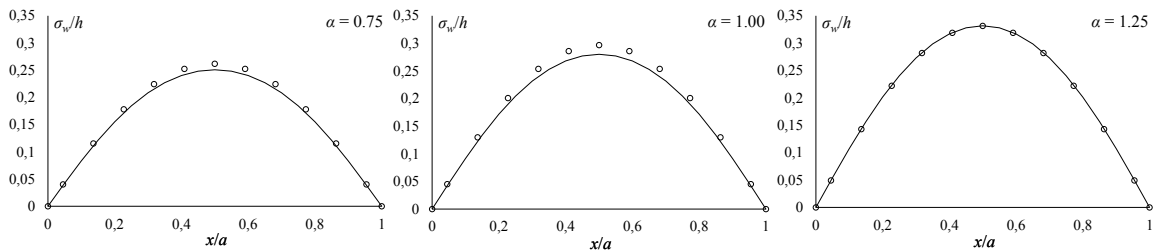


Fig. 2. Standard deviation of the vertical displacement of a rectangular plate along the section  $y = b/2$  for various fractional derivative orders. Continuous lines: SL; dotted lines: BEM.

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## References

- [1] I Podlubny, Fractional Differential Equations: An Introduction to Fractional Derivatives, Fractional Differential Equations, to Methods of Their Solution and Some of Their Applications, Academic Press, San Diego, CA, USA, 1998.
- [2] PG Nutting, A new general law of deformation, Journal of The Franklin Institute-engineering and Applied Mathematics 191 (1921) 679-85.
- [3] A Gemant, A method of analyzing experimental results obtained from elasto-viscous bodies, Physics 7 (1936) 311-7.
- [4] GW Scott-Blair, JE Gaffyn, An application of the theory of quasi-properties to the treatment of anomalous strain-stress relations, Philosophical Magazine Series 7 40 (1949) 80-94.
- [5] RL Bagley, PJ Torvik, A Theoretical Basis for the Application of Fractional Calculus to Viscoelasticity, Journal of Rheology 27 (1983) 201-10.
- [6] YA Rossikhin, MV Shitikova, Applications of Fractional Calculus to Dynamic Problems of Linear and Nonlinear Hereditary Mechanics of Solids, Applied Mechanics Reviews 50 (1997) 15-67.
- [7] YA Rossikhin, MV Shitikova, Application of Fractional Calculus for Dynamic Problems of Solid Mechanics: Novel Trends and Recent Results Applied Mechanics Reviews 63 (2010).
- [8] PD Spanos, BA Zeldin, Random vibration of systems with frequency-dependent parameters or fractional derivatives, Journal of Engineering Mechanics 123 (1997) 290-2.
- [9] OP Agrawal, Stochastic analysis of dynamic systems containing fractional derivatives, Journal of Sound and Vibration 247 (2001) 927-38.
- [10] M Di Paola, G Failla, A Pirrotta, Stationary and non-stationary stochastic response of linear fractional viscoelastic systems, Probabilistic Engineering Mechanics 28 (2012) 85-90.
- [11] PD Spanos, GI Evangelatos, Response of a non-linear system with restoring forces governed by fractional derivatives-Time domain simulation and statistical linearization solution, Soil Dynamics and Earthquake Engineering 30 (2010) 811-21.
- [12] ZL Huang, XL Jin, Response and stability of a SDOF strongly nonlinear stochastic system with light damping modeled by a fractional derivative, Journal of Sound and Vibration 319 (2009) 1121-35.
- [13] A Di Matteo, IA Kougioumtzoglou, A Pirrotta, PD Spanos, M Di Paola, Stochastic response determination of nonlinear oscillators with fractional derivatives elements via the Wiener path integral, Probabilistic Engineering Mechanics 38 (2014) 127-35.
- [14] IA Kougioumtzoglou, PD Spanos, Harmonic wavelets based response evolutionary power spectrum determination of linear and non-linear oscillators with fractional derivative elements, International Journal of Non-Linear Mechanics 80 (2016) 66-75.
- [15] Y Yang, W Xu, W Jia, Q Han, Stationary response of nonlinear system with Caputo-type fractional derivative damping under Gaussian white noise excitation, Nonlinear Dynamics 79 (2015) 139-46.
- [16] Z Zheng-you, L Gen-guo, C Chang-jun, Quasi-static and dynamical analysis for viscoelastic Timoshenko beam with fractional derivative constitutive relation, Appl Math Mech 23 (2002) 1-12.
- [17] Z-f Liang, X-y Tang, Analytical solution of fractionally damped beam by Adomian decomposition method, Appl Math Mech 28 (2007) 219-28.
- [18] G Failla, Stationary Response of Beams and Frames with Fractional Dampers through Exact Frequency Response Functions, Journal of Engineering Mechanics (2016) D4016004.
- [19] G-G Li, Z-Y Zhu, C-J Cheng, Application of Galerkin Method to Dynamical Behavior of Viscoelastic Timoshenko Beam with Finite Deformation, Mech Time-Depend Mater 7 (2003) 175-88.
- [20] YA Rossikhin, MV Shitikova, The analysis of the impact response of a thin plate via fractional derivative standard linear solid model, Journal of Sound and Vibration 330 (2011) 1985-2003.
- [21] Y Rossikhin, M Shitikova, Nonlinear dynamic response of a thin plate embedded in a fractional viscoelastic medium under combinational internal resonances, Applied Mechanics and Materials 595 (2014) 105-10.
- [22] YA Rossikhin, MV Shitikova, JC Ngenzi, A new approach for studying nonlinear dynamic response of a thin plate with internal resonance in a fractional viscoelastic medium, Shock and Vibration 2015 (2015).
- [23] M Abramowitz, IA Stegun, Handbook of Mathematical Functions. With Formulas, Graphs, and Mathematical Tables. with Corr, U.S. Department of Commerce National Bureau of Standards, 1968.
- [24] AK Grünwald, Ueber "begrenzte" derivationen und deren anwendung, Zeitschrift f Mathematik u Physik 12 (1867) 441-80.
- [25] JT Katsikadelis, MS Nerantzaki, Non-linear analysis of plates by the Analog Equation Method, Computational Mechanics 14 (1994) 154-64.